VCE UNITS 3 & 4 · EXAM REFERENCE

Mathematical Methods 3 & 4

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01 Graph Sketching

The standard form and key features of every graph family

To sketch a graph, identify all of its key features: the axis intercepts, any asymptotes, any key points such as a turning point, a point of inflection, or the centre of a shape. For circular functions: find its period, amplitude, and asymptotes if applicable

I Linear

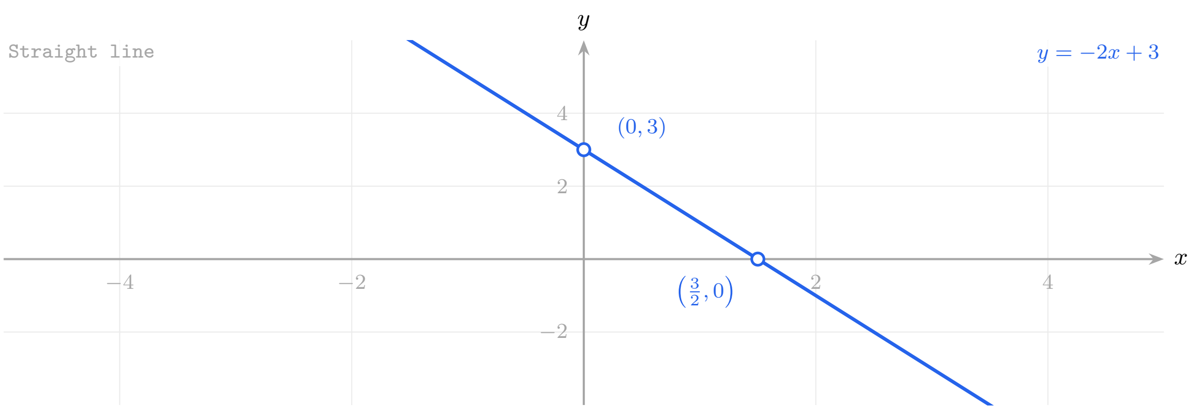
A straight line, fixed by a gradient and a starting height.

FORM $y = mx + c$

m is the gradient, c is the y intercept

Features

- **Gradient** $m = \frac{\text{rise}}{\text{run}} = \frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta = \frac{\sin(\theta)}{\cos(\theta)}$
- y **intercept** at $(0, c)$, set $x = 0$
- x **intercept** at $(-\frac{c}{m}, 0)$, set $y = 0$
- **Angle to the** x **axis:** $\theta = \tan^{-1}(m)$



EXAMPLE *Sketch* $y = -2x + 3$

y intercept $= 3$. x intercept: $0 = -2x + 3 \Rightarrow x = \frac{3}{2}$.
The gradient is negative, so $\tan\theta = -2$ and the line makes an obtuse angle $\theta = 180° - \tan^{-1}(2) \approx 116.6°$ with the positive x axis.

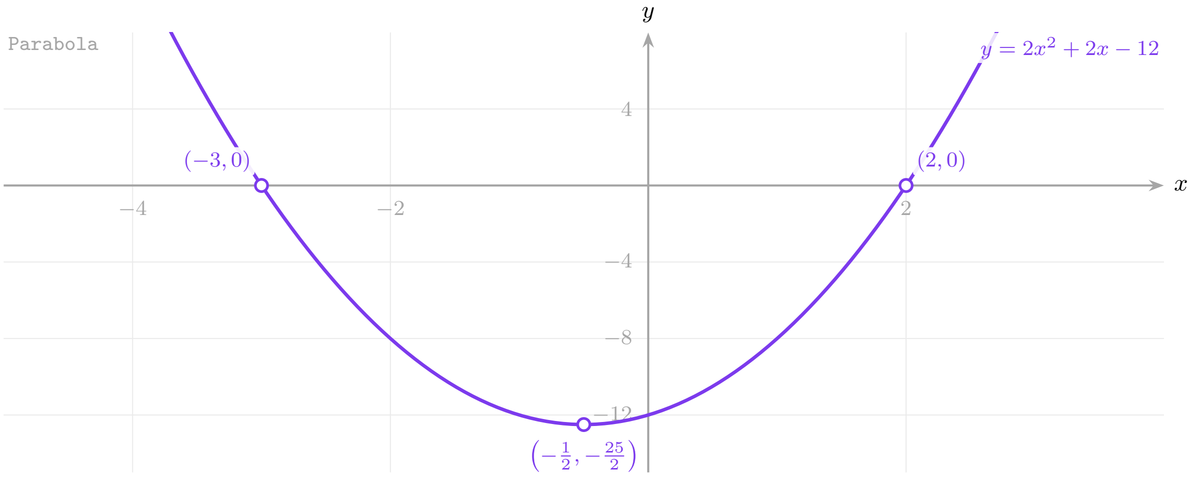
II Quadratic

The three forms below describe the same curve $y = 2x^2 + 2x - 12$; each reads off a different feature.

GENERAL	$y = 2x^2 + 2x - 12$	y intercept -12
FACTORISED	$y = 2(x + 3)(x - 2)$	x intercepts -3, 2
TURNING POINT	$y = 2\left(x + \frac{1}{2}\right)^2 - \frac{25}{2}$	TP $(-\frac{1}{2}, -\frac{25}{2})$

The four quadratic tools

- **Roots** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- **Axis of symmetry** $x = -\dfrac{b}{2a}$
- **Discriminant** $\Delta = b^2 - 4ac$: $\Delta > 0$ two solutions, $\Delta = 0$ one solution , $\Delta < 0$ no Real solutions
- **Turning point** where $\dfrac{dy}{dx} = 0$



EXAMPLE *Sketch* $y = 2x^2 + 2x - 12$

$$y = 2(x^2 + x - 6) = 2\left(x + \tfrac{1}{2}\right)^2 - \tfrac{25}{2} = 2(x+3)(x-2)$$

Axis of symmetry $x = -\frac{b}{2a} = -\frac{1}{2}$, giving the turning point $\left(-\frac{1}{2}, -\frac{25}{2}\right)$, with x intercepts at $x = -3$ and $x = 2$.

III Cubic

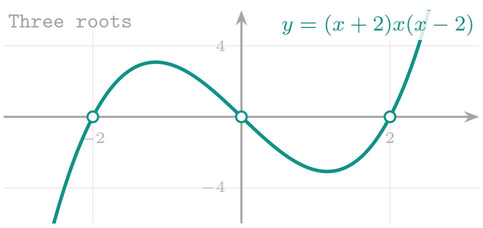
Up to three roots and a point of inflection where the bend reverses.

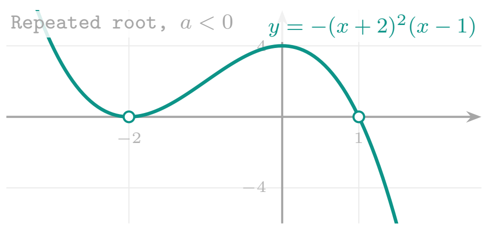
GENERAL $y = ax^3 + bx^2 + cx + d$

INFLECTION $y = a(x-h)^3 + k$

FACTORISED $y = a(x - r_1)(x - r_2)(x - r_3)$

three x intercepts





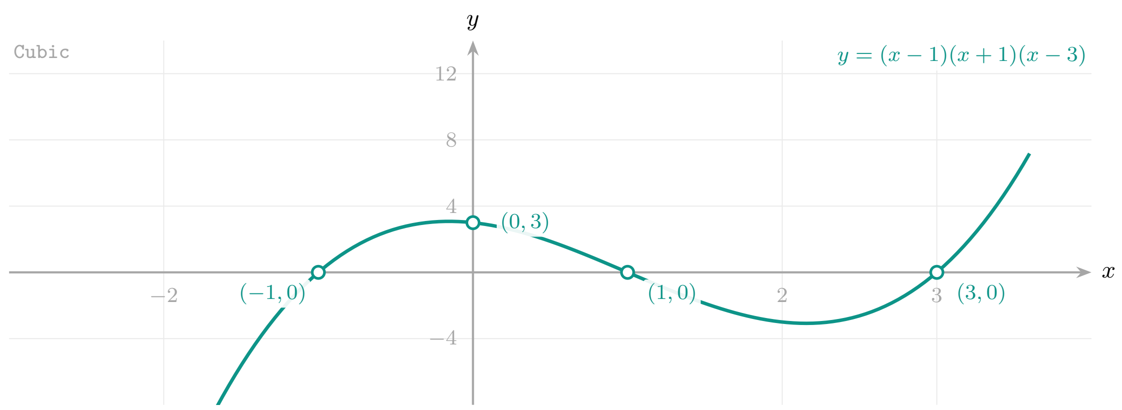
Steps to sketch a cubic

- Factorise: use the factor theorem for the first root, then long division for the rest.
- Three distinct roots means the curve crosses three times; a repeated root means it touches there.
- Mark the y intercept and any stationary points, then join with the correct end behaviour.

EXAMPLE *Sketch* $y = x^3 - 3x^2 - x + 3$

$$y = x^2(x-3) - (x-3) = (x-3)(x^2-1) = (x-3)(x-1)(x+1)$$

Roots at $x = -1, 1, 3$ and y intercept $(0, 3)$. The leading coefficient is positive, so the curve rises to the right.



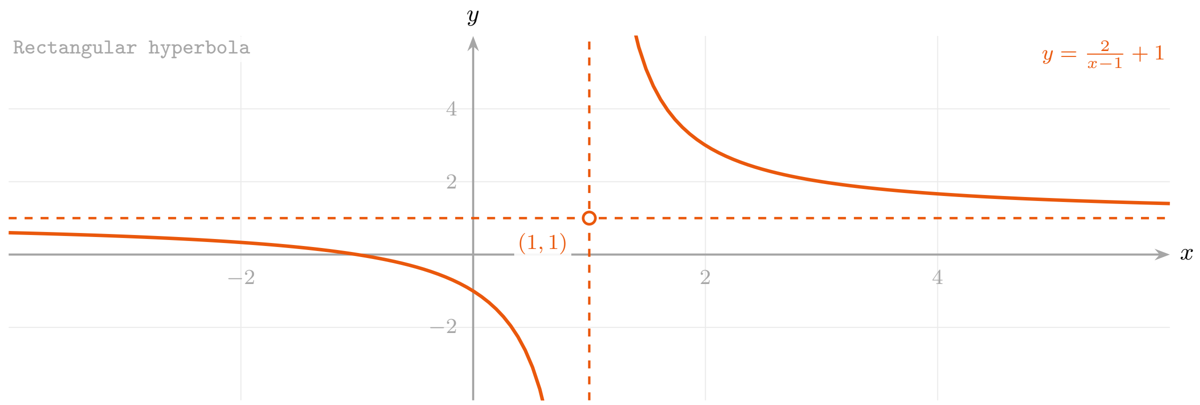
IV Hyperbola

Two branches sweeping toward a vertical and a horizontal asymptote that cross at the centre.

FORM $y = \dfrac{a}{x-h} + k$

Features

- **Vertical asymptote** $x = h$, where the denominator is 0
- **Horizontal asymptote** $y = k$; **centre** (h, k)
- **Sign of** a: $a > 0$ places branches top right and bottom left; $a < 0$ flips them



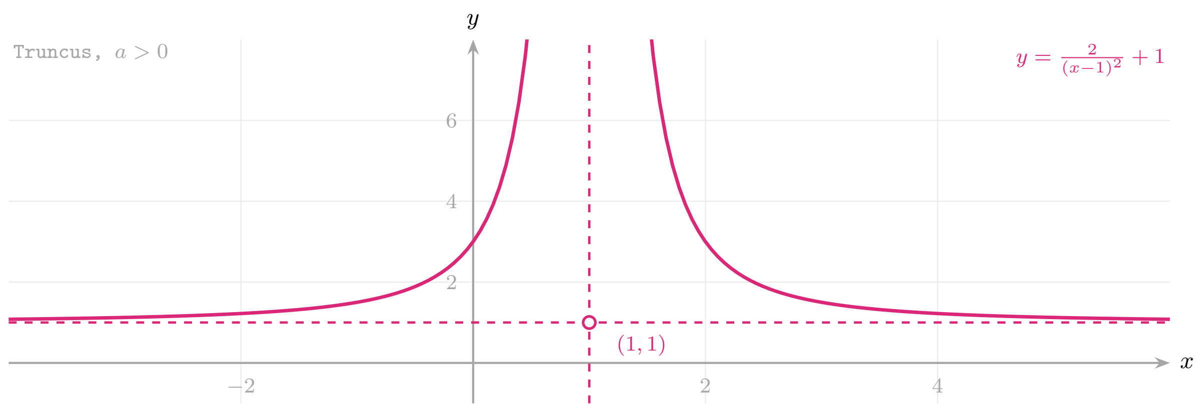
V Truncus

A hyperbola with a squared denominator, so both branches sit on the same side of the horizontal asymptote.

FORM $y = \dfrac{a}{(x-h)^2} + k$

Features

- **Vertical asymptote** $x = h$, from $(x-h)^2 = 0$
- **Horizontal asymptote** $y = k$; **centre** (h, k)
- **Sign of** a: $a > 0$ both branches above $y = k$; $a < 0$ both below

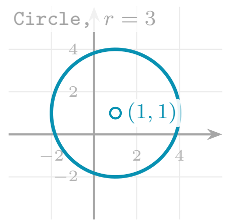


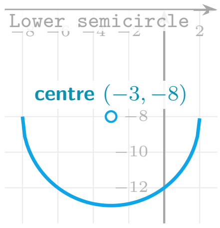
VI Circle and semicircle

A full circle takes two values of y for each x; a semicircle keeps one half using a $\pm$ sign.

CIRCLE	$(x-h)^2 + (y-k)^2 = r^2$	centre (h, k), radius r
SEMICIRCLE	$y = \pm\sqrt{r^2 - (x-h)^2} + k$	top or bottom half

Complete the square to reveal the centre and radius. For $x^2 - 6x + y^2 - 10y = 225$ this gives $(x-3)^2 + (y-5)^2 = 259$, centre $(3, 5)$, radius $\sqrt{259}$. Left and right semicircles swap x and y: $x = \pm\sqrt{r^2 - (y-k)^2} + h$.

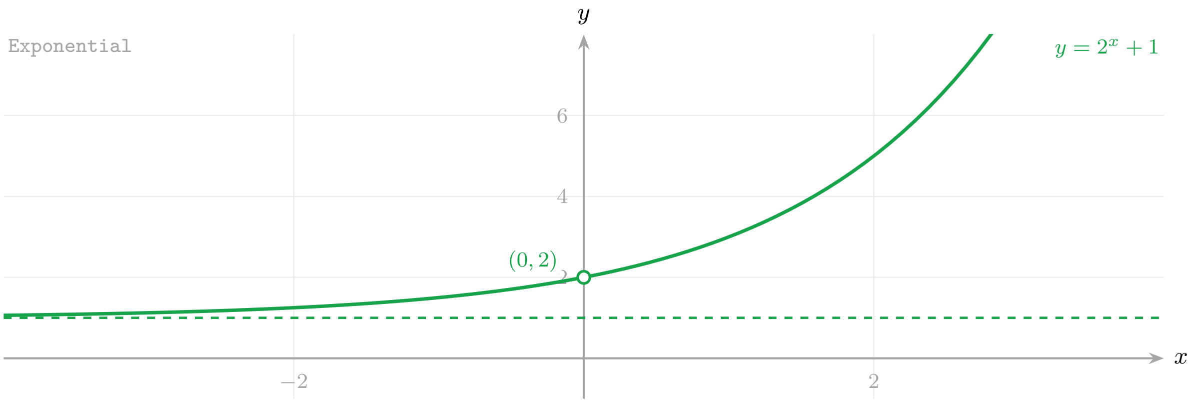




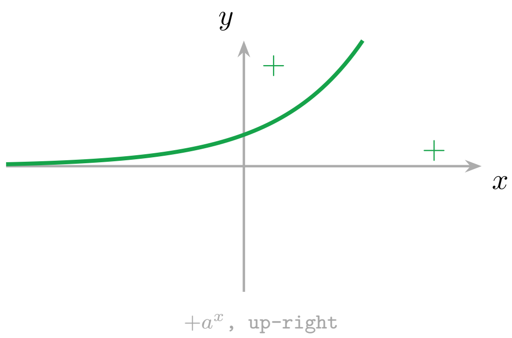
VII Exponential

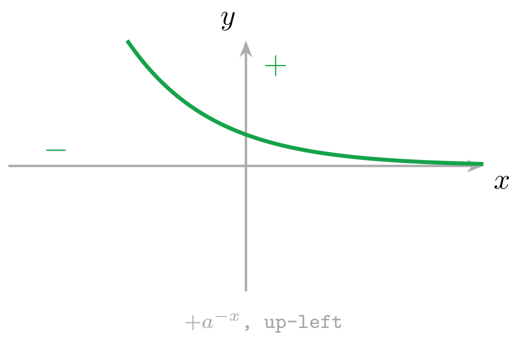
The direction follows from the two signs, one on the function and one on the power.

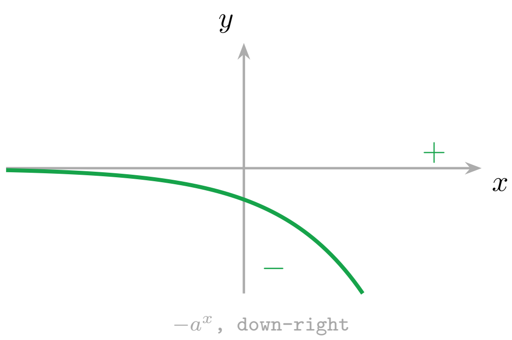
FORM $y = \pm a^{\pm x} + k$ horizontal asymptote $y = k$

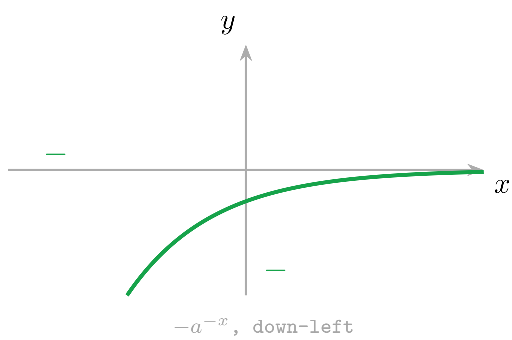


Direction from the two signs









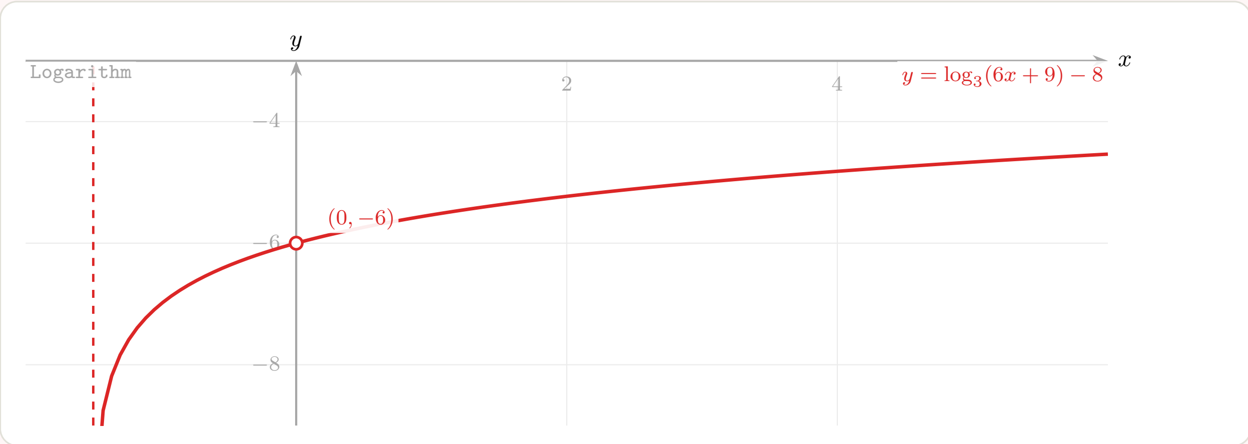
VIII Logarithm

The inverse of the exponential, reflected in $y = x$. A vertical asymptote and no horizontal one.

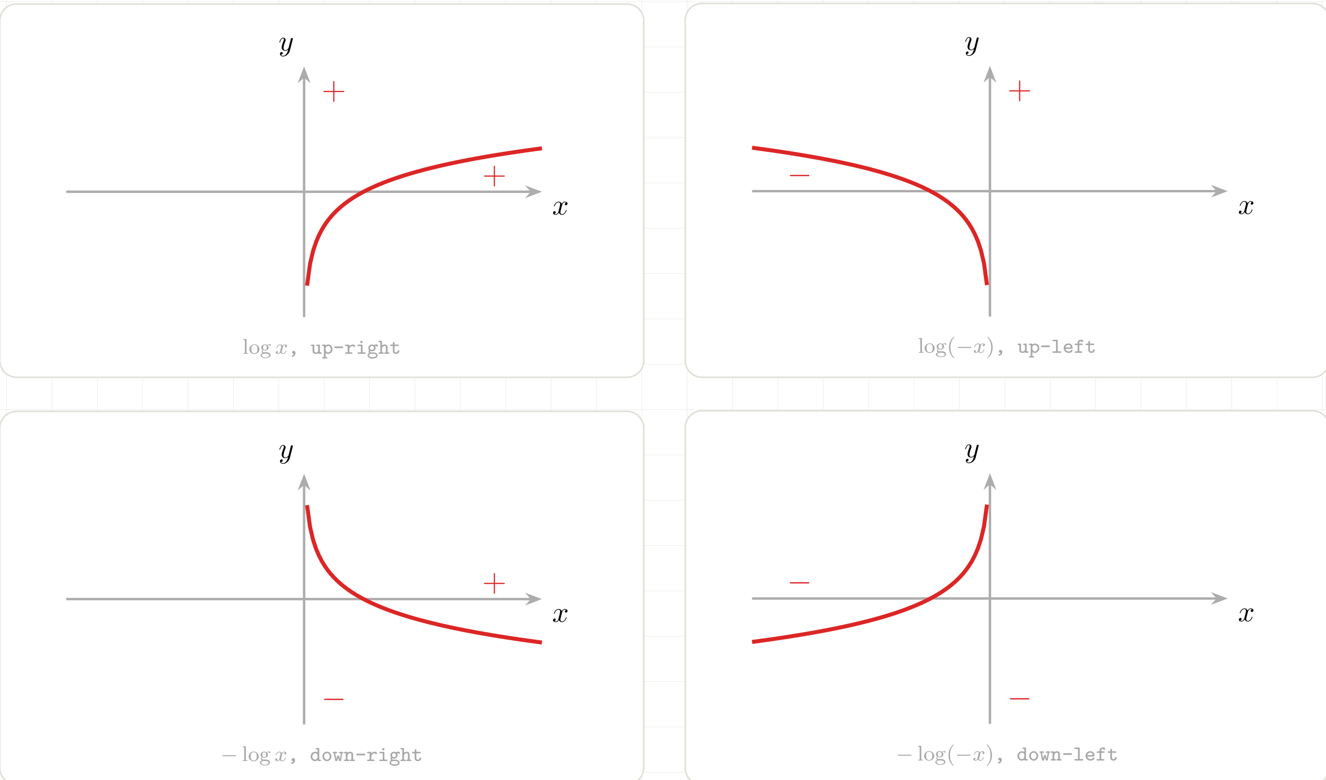
FORM $y = \log_a(bx + c) + k$ vertical asymptote where $bx + c = 0$

EXAMPLE *Sketch* $y = \log_3(6x + 9) - 8$

- **Vertical asymptote**: $6x + 9 = 0 \Rightarrow x = -\frac{3}{2}$
- x **intercept** $(y = 0)$: $\log_3(6x + 9) = 8 \Rightarrow 6x + 9 = 3^8 = 6561 \Rightarrow x = 1092$
- y **intercept** $(x = 0)$: $y = \log_3 9 - 8 = 2 - 8 = -6$



Direction from the two signs



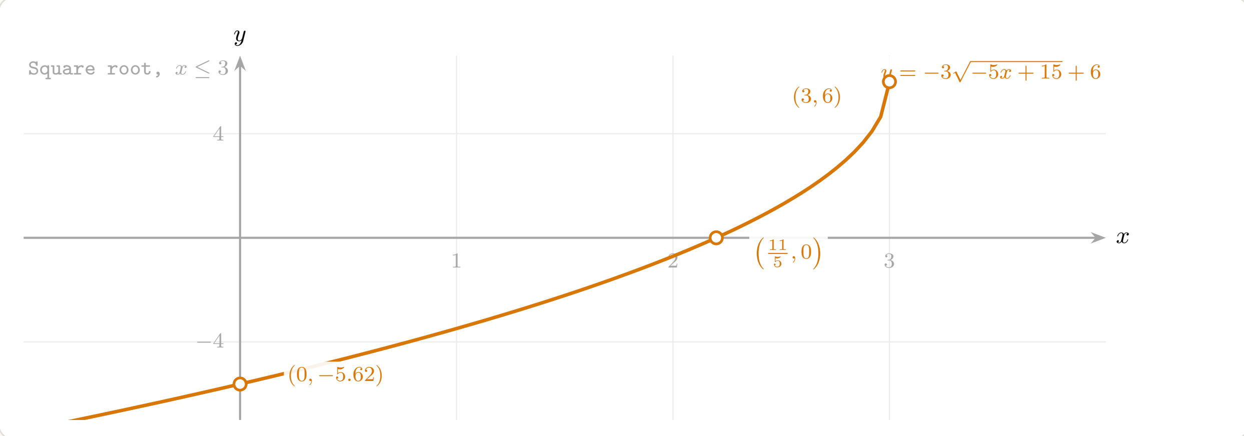
IX Square root

The inverse of the quadratic. No asymptotes, but a definite starting point.

FORM $y = n\sqrt{ax + b} + c$ starting point where $ax + b = 0$

EXAMPLE *Sketch* $y = -3\sqrt{-5x+15}+6$, *domain* $x \le 3$

- **Starting point** (root $= 0$): $-5x+15=0 \Rightarrow x=3$, so $(3, 6)$
- x **intercept**: $\sqrt{-5x+15}=2 \Rightarrow x=\frac{11}{5}$, so $\left(\frac{11}{5}, 0\right)$
- y **intercept**: $y = 6-3\sqrt{15} \approx -5.62$, so $(0, -5.62)$



02 Composite Functions

The RIDO condition and domain restriction

A composite function is a function evaluated inside another. Given $f(x) = 3x^2 - 5$ and $g(x) = \log_e(x)$, the composite is $g(f(x)) = \log_e(3x^2 - 5)$.

The RIDO condition

For a composite to exist, the **R**ange of the **I**nside must lie within the **D**omain of the **O**utside. Check RIDO before anything else.

$g(f(x))$ does not exist

f has range $[-5, \infty)$; $g = \log_e(x)$ has domain $(0, \infty)$.

Since $[-5, \infty) \nsubseteq (0, \infty)$, RIDO fails.

For instance $f(-1) = -2$, and $g(-2) = \log_e(-2)$ is undefined.

$f(g(x))$ exists

g has range $\mathbb{R}$; f has domain $\mathbb{R}$.

Since $\mathbb{R} \subseteq \mathbb{R}$, RIDO holds, so $f(g(x)) = 3(\log_e x)^2 - 5$ is defined.

Making a composite exist by restricting the domain

If RIDO fails, restrict the domain of the inside function so its range shrinks to fit the domain of the outside.

EXAMPLE *Restrict f so that $g(f(x))$ exists*

The range of f must sit within $(0, \infty) = \operatorname{dom} g$, so require $3x^2 - 5 > 0$:

$$x^2 > \frac{5}{3} \;\Rightarrow\; x > \sqrt{\frac{5}{3}} \approx 1.291$$

Restricting to $f_1 : (1.291, \infty) \to \mathbb{R},\ f_1(x) = 3x^2 - 5$ makes $g(f_1(x))$ exist.

Two things to nail

1. Identify clearly the Domain and Range of each function f and g. Do not confuse them.
2. The domain of a composite equals the domain of the inside function, that is $\operatorname{dom}(f \circ g) = \operatorname{dom} g$.

03 Trigonometry

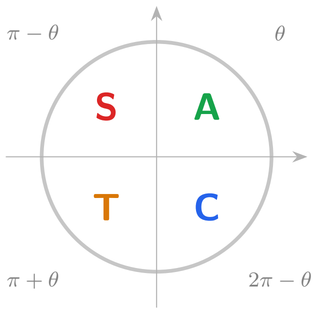
Exact values, symmetry and general solutions

The exact value table, the ASTC symmetry circle and the general solutions carry most trigonometry questions. Sketching then follows the same method as any other graph.

1 · Exact values for the five basic angles

Degrees	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	undefined

2 · Symmetry, using ASTC



The four reflections

- **Q2** $\pi-\theta$: $\sin +,\ \cos -,\ \tan -$
- **Q3** $\pi+\theta$: $\sin -,\ \cos -,\ \tan +$
- **Q4** $2\pi-\theta$: $\sin -,\ \cos +,\ \tan -$

e.g. $\sin(\pi-\theta)=\sin\theta$, $\cos(\pi+\theta)=-\cos\theta$, $\tan(2\pi-\theta)=-\tan\theta$.

3 · General solutions

$\boxed{\cos x = a}$ $x = 2n\pi \pm \cos^{-1}(a)$ $a\in[-1,1]$

$\boxed{\tan x = a}$ $x = n\pi + \tan^{-1}(a)$ $a\in\mathbb{R}$

$\boxed{\sin x = a}$ $x = 2n\pi + \sin^{-1}(a)$ or $(2n+1)\pi - \sin^{-1}(a)$ $a\in[-1,1]$

In every case $n\in\mathbb{Z}$.

4 · Sketching sine and cosine

For $y = a\sin(n(x+\theta))+b$ on domain $[D, M]$, always find the amplitude, period, mean line, starting point and endpoint.

The checklist

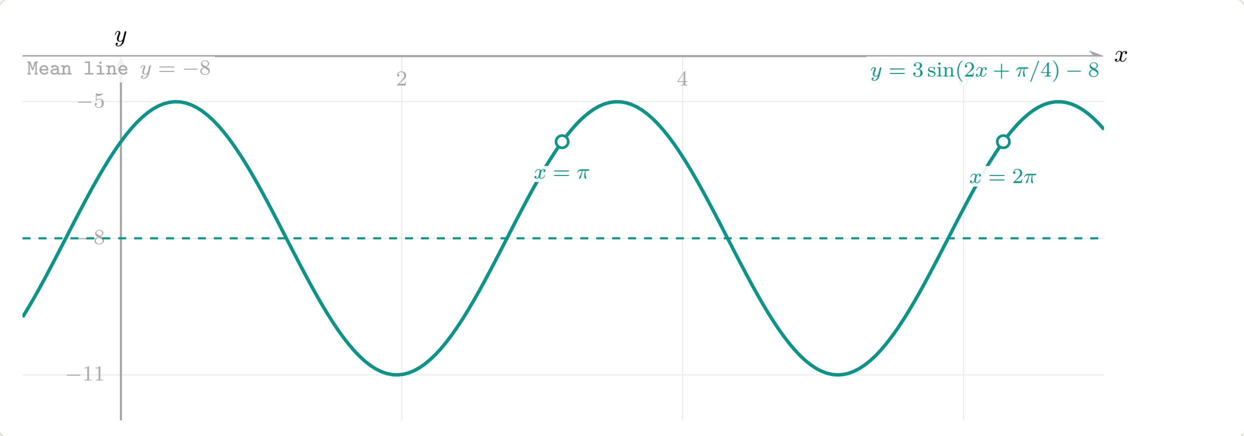
- **Amplitude** $= |a|$
- **Period** $= \dfrac{2\pi}{n}$
- **Mean line** $y = b$; **range** $[b - a,\ b + a]$
- **Starting point**: set $n(x + \theta) = 0$
- **Endpoint** $=$ starting point $+$ period

5-point Method:

For both sin and cos graphs, one cycle contains 5 points where Starting point is the first point, and Endpoint is the 5th one. Find the 3rd point which is in the middle of those 2. And find the other 2 points which are in the middle of the 2 halves created by the current 3 points. Sketch one cycle first, then repeat it across the domain and read off the real endpoints and intercepts.

EXAMPLE *Sketch* $y = 3\sin(2x + \frac{\pi}{4}) - 8$, *domain* $[\pi, 2\pi]$

Amplitude 3, period $\frac{2\pi}{2} = \pi$. Starting point: $2x + \frac{\pi}{4} = 0 \Rightarrow x = -\frac{\pi}{8}$, giving $\left(-\frac{\pi}{8}, -8\right)$, so the endpoint is $\left(\frac{7\pi}{8}, -8\right)$. At the domain ends, $y = \frac{3\sqrt{2}}{2} - 8 \approx -5.88$.



5 · Sketching tangent

For $y = a\tan(n(x + \theta)) + b$ on domain $[D, M]$:

Steps

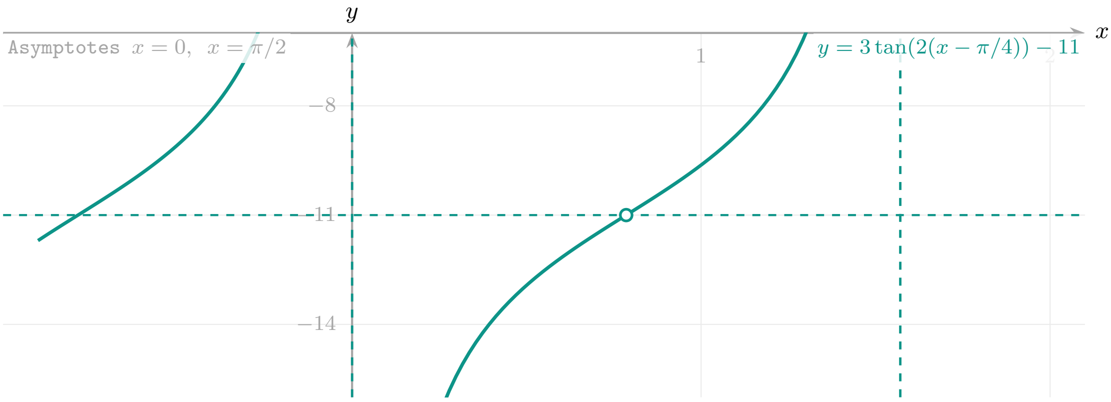
- **Period** $= \dfrac{\pi}{n}$, the distance between consecutive asymptotes.
- **Asymptotes**: tan is undefined at $\frac{\pi}{2}$, so set $n(x+\theta) = \frac{\pi}{2}$ and solve for the first asymptote. Find the others by adding and subtracting the period.
- $\pm a$ gives the points halfway between each mean line crossing $(y = b)$ and the asymptote.
- Find endpoints and axis intercepts if required.

EXAMPLE *Sketch* $y = 3\tan(2(x - \frac{\pi}{4})) - 11$, *domain* $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$

Period $= \frac{\pi}{2}$. First asymptote: set the argument equal to $\frac{\pi}{2}$:

$$2\left(x - \tfrac{\pi}{4}\right) = \tfrac{\pi}{2} \;\Rightarrow\; x - \tfrac{\pi}{4} = \tfrac{\pi}{4} \;\Rightarrow\; x = \tfrac{\pi}{2}$$

Add and subtract the period $\frac{\pi}{2}$ to get the neighbouring asymptotes $x = 0$ and $x = \pi$. Within the domain the asymptote is $x = 0$, and the mean line $y = -11$ is crossed at the centre $x = \frac{\pi}{4}$.



y
x
Asymptotes $x = 0$, $x = \pi/2$
$y = 3\tan(2(x - \pi/4)) - 11$
1
2
-8
-11
-14

04 Transformations

Applying a transformation and deriving one

There are two question types, and they are inverses of one another. One applies a transformation to produce a new rule; the other reads a transformation off two given equations.

Type 1 · apply a transformation

Make x and y the subject in terms of x' and y', then substitute into the original $y(x)$ to obtain the new rule $y'(x')$.

The transformed rule must contain x' and y'.

Type 2 · derive a transformation

Make x' and y' the subject, then read the transformations directly from those expressions.

The original rule must contain x and y.

Watch the axis swap

For type 2 questions - deriving the transformation, dilation or reflection written on x acts on the y values and vice versa. A translation stays on its own axis, and a plain number is unchanged.

Function notation summary

Translations	Reflections	Dilations
$f(x-h)$: right h	$f(-x)$: in y axis	$f(ax)$: factor $\frac{1}{a}$ from y axis
$f(x)+k$: up k	$-f(x)$: in x axis	$a\,f(x)$: factor a from x axis

EXAMPLE *Type 1 · apply a transformation to $y = x^2$*

Apply a dilation by factor 2 from the x axis, then a translation 3 right and 1 up. The mapping is

$$x' = x + 3, \qquad y' = 2y + 1.$$

Make x and y the subject and substitute into $y = x^2$:

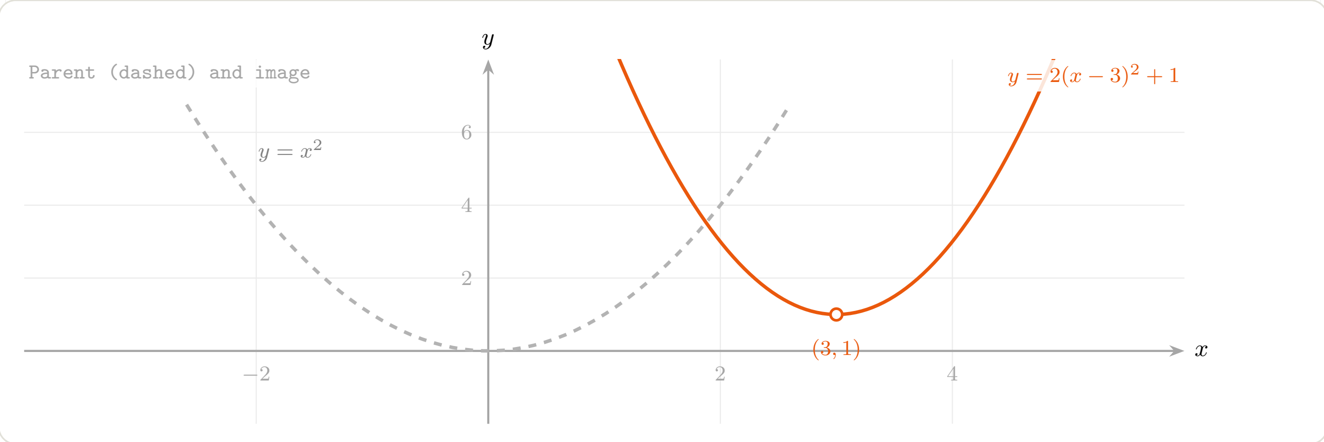
$$x = x' - 3, \quad y = \frac{y'-1}{2} \;\Rightarrow\; \frac{y'-1}{2} = (x'-3)^2 \;\Rightarrow\; y' = 2(x'-3)^2 + 1.$$

EXAMPLE *Type 2 · derive the transformation from $y = x^2$ to $y = 2(x-3)^2 + 1$*

Compare the image $y' = 2(x'-3)^2 + 1$ with $y = x^2$, so that x plays the role of $x' - 3$ and y the role of $\frac{y'-1}{2}$:

$$x = x' - 3 \Rightarrow x' = x + 3, \qquad y = \frac{y'-1}{2} \Rightarrow y' = 2y + 1.$$

Reading these off: a dilation by factor 2 from the x axis (the axis swap), a translation 3 right and 1 up. This is exactly the reverse of Type 1.



Interactive version

An interactive version of this diagram, where you drag a, h and k and watch the curve update, is available in the web edition of this reference.

05 Calculus

Differentiation, integration and their applications

The differentiation and integration basic formulas, followed by the applications that carry the exam: tangents and normals, derivative graphs, area estimation and motion.

Derivatives and antiderivatives

DIFFERENTIATE	INTEGRATE
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$
$\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$	$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$
$\frac{d}{dx}e^{ax} = ae^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\log_e x = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \log_e x + c,\ x > 0$
$\frac{d}{dx}\log_e(ax+b) = \frac{a}{ax+b}$	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + c$
$\frac{d}{dx}\sin ax = a\cos ax$	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$
$\frac{d}{dx}\cos ax = -a\sin ax$	$\int \cos ax\,dx = \frac{1}{a}\sin ax + c$
$\frac{d}{dx}\tan ax = \frac{a}{\cos^2 ax} = a\sec^2 ax$	$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$

Differential Rules

- **Product** $(uv)' = u\frac{dv}{dx} + v\frac{du}{dx}$
- **Quotient** $\left(\frac{u}{v}\right)' = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- **Chain** $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Numerical methods

Estimating roots

Bisection: for a root in $[a, b]$ with $f(a) * f(b) < 0$, take $c = \frac{a+b}{2}$ and repeat on the half that still contains the root.

Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Estimating area (strip width w)

Left-endpoint $A \approx w[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$

Right-endpoint $A \approx w[f(x_1) + f(x_2) + \cdots + f(x_n)]$

Trapezium $A \approx \frac{w}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$

Properties of the definite integral

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

The **average value** on $[a, b] = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

Rates of change, tangents and normals

Average and instantaneous

The **average rate** over $[a, b]$ is the chord gradient $\dfrac{y_2 - y_1}{x_2 - x_1}$.

The **instantaneous rate** at $x = x_1$ is $f'(x_1)$, the tangent gradient.

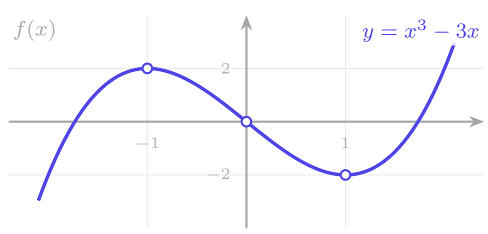
At a point (x_1, y_1)

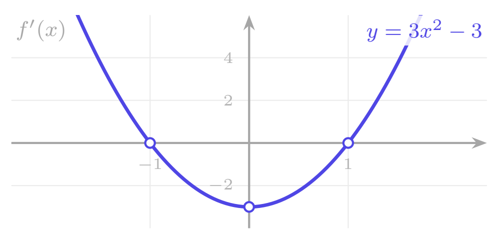
Tangent $y - y_1 = f'(x_1)(x - x_1)$

Normal $y - y_1 = \dfrac{-1}{f'(x_1)}(x - x_1)$

The derivative graph

Moving from f to f' lowers the degree by one; moving back raises it. Track how the features correspond.





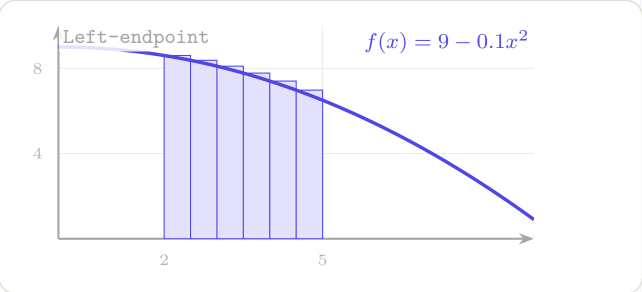
$f \to f'$ lowers the degree

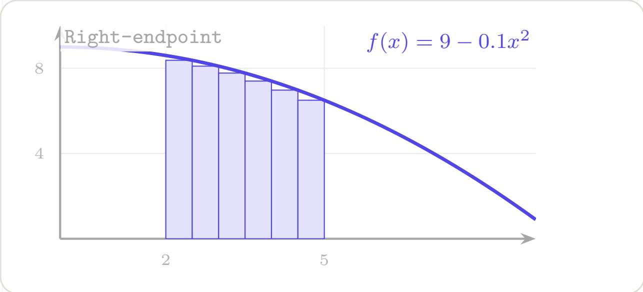
- Turning point of $f \to x$ intercept of f'
- Point of inflection of $f \to$ turning point of f'
- Sharp corner of $f \to$ break in f'

$f' \to f$ raises the degree

- x intercept of $f' \to$ turning point of f
- Turning point of $f' \to$ point of inflection of f

Estimating area under a graph





For a decreasing graph the left endpoint estimate is too large and the right endpoint estimate is too small (the reverse holds for an increasing graph). Each rectangle area is width times height, and the estimate is their sum.

Integration by recognition

Method

- Differentiate the given expression to obtain $A' = B$.
- Reverse it: $\int B\,dx = A$.
- Rearrange to the integral required.

EXAMPLE *Find* $\dfrac{d}{dx}\log_e(2x^3-1)$, *hence* $\displaystyle\int \frac{x^2}{2x^3-1}\,dx$

$$\left(\log_e(2x^3-1)\right)' = \frac{6x^2}{2x^3-1} \;\Rightarrow\; \int \frac{6x^2}{2x^3-1}\,dx = \log_e(2x^3-1)$$

$$\therefore \int \frac{x^2}{2x^3-1}\,dx = \tfrac{1}{6}\log_e(2x^3-1) + c$$

Motion in a straight line

The chain of derivatives

position x differentiates to velocity v, which differentiates to acceleration a. Integrate to reverse each step. Speed is $|v|$.

EXAMPLE *A body and its velocity*

A body starts at O and moves in a straight line. After t seconds $(t \geq 0)$ its velocity is $v = 2t - 4$ m/s. Find (a) its position x, (b) its position after 3 seconds, (c) the distance travelled in the first 3 seconds, and (d) its average speed over the first 3 seconds.

(a) $x = \displaystyle\int v\,dt = t^2 - 4t + c$; it starts at O, so $c = 0$ and $x = t^2 - 4t$.

(b) At $t = 3$, $x = 9 - 12 = -3$ m.

(c) The body reverses when $v = 0$, that is $t = 2$, where $x = -4$. It travels $0 \to -4$ (distance 4) then $-4 \to -3$ (distance 1), giving 5 m.

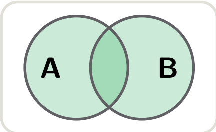
(d) Average speed $= \dfrac{\text{distance}}{\text{time}} = \dfrac{5}{3}$ m/s. The average velocity uses displacement: $\dfrac{-3}{3} = -1$ m/s.

06 Probability

Discrete and continuous variables; Random sampling and Confidence Interval

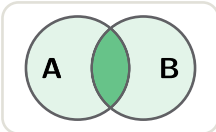
I · Basic probability

1 · Additional rule



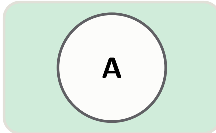
$\Pr(A \cup B) =$
$\Pr(A) + \Pr(B) - \Pr(A \cap B)$

2 · Conditional rule



$\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

3 · Complement rule



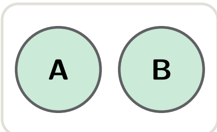
$\Pr(A') = 1 - \Pr(A)$;
$\Pr(A' \cap B') = 1 - \Pr(A \cup B)$
(1− *FlipEverything* Rule)

4 · Total probability

$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$, read across a row of the two-way table:

	B	B'	
A	$A \cap B$	$A \cap B'$	A
A'	$A' \cap B$	$A' \cap B'$	A'
	B	B'	1

Mutually exclusive



The events cannot occur together, so $\Pr(A \cap B) = 0$ and $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Independent

One event does not affect the other, so $\Pr(A|B) = \Pr(A)$ and $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

Do not confuse the two. Mutually exclusive means the events never happen together; independent means one has no effect on the other. Mutually exclusive events with non zero probability are never independent.

Counting

Combinations (order does not matter): ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, with ${}^nC_0 = {}^nC_n = 1$, ${}^nC_1 = n$ and ${}^nC_r = {}^nC_{n-r}$.

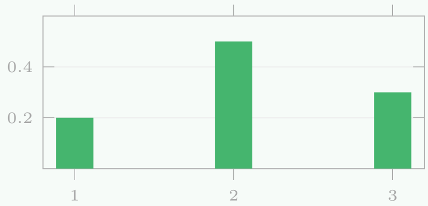
Permutations (order matters): ${}^nP_r = \dfrac{n!}{(n-r)!}$.

II · Discrete probability

Two kinds appear: a random discrete variable given by a probability table, and the binomial distribution.

EXAMPLE *Random discrete variable with a table*

x	1	2	3
$p(x)$	0.2	0.5	0.3



$$\mu = E(X) = \sum x\,p(x) = 1(0.2) + 2(0.5) + 3(0.3) = 2.1$$

$$E(X^2) = 1(0.2) + 4(0.5) + 9(0.3) = 4.9$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 4.9 - 2.1^2 = 0.49$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = 0.7$$

Binomial: three conditions

- Each trial has two outcomes, success p or failure $1-p$.
- A fixed number n of independent trials.
- Constant success probability p.

Binomial: variables and formula

n trials, success probability p, x successes.

$$\Pr(X = x) = {}^nC_x\,p^x(1-p)^{n-x}$$

$\mu = np$, $\mathrm{Var}(X) = np(1-p)$, $\sigma = \sqrt{np(1-p)}$.

Inverse binomial on the TI-Nspire

When the unknown is n (the smallest number of trials for a required probability), use `Menu 5-5-D` invBinomN, or define $b(n) =$binomCdf and scan a spreadsheet for the first n that meets the condition.

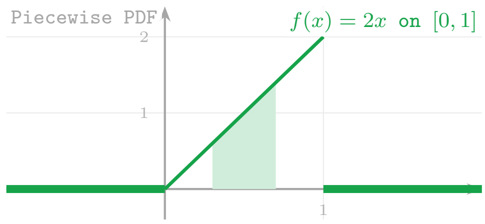
III · Continuous probability

Two kinds appear: a random continuous variable given by a probability density function (PDF), and the normal distribution.

Two conditions for a PDF

- $f(x) \geq 0$ for all x.
- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Probability is area under the curve, $\Pr(a \leq X \leq b) = \displaystyle\int_a^b f(x)\,dx$.



Mean, variance, median and IQR

$$\mu = E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx, \qquad E(X^2) = \int_{-\infty}^{\infty} x^2\,f(x)\,dx$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2, \qquad \sigma = \sqrt{\mathrm{Var}(X)}$$

$$\text{median } m : \int_{-\infty}^{m} f(x)\,dx = 0.5, \qquad \text{IQR} = Q_3 - Q_1$$

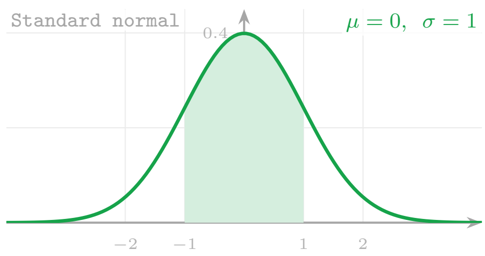
where Q_3, Q_1 satisfy $\displaystyle\int_{-\infty}^{Q_3} f\,dx = 0.75$ and $\displaystyle\int_{-\infty}^{Q_1} f\,dx = 0.25$.

When you sketch a piecewise PDF, draw the $y = 0$ pieces on the axis as well, so the whole domain is shown.

The normal bell curve

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Standardised score $z = \dfrac{x-\mu}{\sigma}$. About 68% of values lie within 1σ, 95% within 2σ, 99.7% within 3σ.



Normal question types, with a worked example of each

GIVEN	FIND	WORKED EXAMPLE
μ, σ, x	a probability	$X \sim N(50, 5^2)$: $\Pr(X < 58) =$ `normCdf` $(-\infty, 58, 50, 5) \approx 0.945$
$\mu, \sigma, \Pr$	the value x	$\Pr(X < k) = 0.9$: $k =$ `invNorm` $(0.9, 50, 5) \approx 56.4$
$\mu, \Pr, x$	the SD σ	$X \sim N(42, \sigma)$, $\Pr(X > 36) = 0.85$: $z =$ `invNorm` $(0.15, 0, 1) = -1.036$, $\sigma = \frac{36-42}{-1.036} \approx 5.79$

For invNorm, always enter the probability to the left, $\Pr(-\infty, x)$.

IV · Random sampling

Notation

- N population size, D desired quantity in the population, n sample size, x desired quantity in the sample.
- **Population proportion** $p = \dfrac{D}{N}$; **sample proportion** $\hat{p} = \dfrac{x}{n}$, which lies in $[0, 1]$.

Large population

$E(\hat{p}) = p$, $\text{sd}(\hat{p}) = \sqrt{\dfrac{p(1-p)}{n}}$.

- N **large,** n **small**: binomial, $x = n\hat{p}$, $X \sim \text{Bi}(n, p)$.
- N **large,** n **large**: normal, $\hat{p} \sim N\left(p, \dfrac{p(1-p)}{n}\right)$.

Small population, hypergeometric

$$\Pr(X = x) = \frac{{}^{D}C_x \; {}^{N-D}C_{n-x}}{{}^{N}C_n}$$

EXAMPLE *Small population: a tank of* 10 *fish,* 7 *gold, draw* $n = 3$ HYPERGEOMETRIC

Here $N = 10$, $D = 7$, $n = 3$, so $\hat{p}$ takes $0, \frac{1}{3}, \frac{2}{3}, 1$.

x	0	1	2	3
$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(X = \hat{p})$	$\frac{1}{120}$	$\frac{7}{40}$	$\frac{21}{40}$	$\frac{7}{24}$

Then $E(\hat{P}) = \frac{7}{10} = p$ and $\text{sd}(\hat{P}) = \sqrt{E(\hat{P}^2) - [E(\hat{P})]^2} = \frac{7}{30}$.

EXAMPLE *Large population, small sample: 70% of 17 year olds attend school, $n = 20$* BINOMIAL APPROXIMATION

$X \sim \text{Bi}(20, 0.7)$, $X = n\hat{p}$.

a $\Pr(\hat{P} = 0.7) = \Pr(X = 14) = {}^{20}C_{14}\,(0.7)^{14}(0.3)^6 = 0.1916$

b $\text{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{0.7(0.3)}{20}} = 0.1025$, so $p \pm \text{sd}$ gives 0.5975 and 0.8025.

$$\Pr(0.5975 \le \hat{P} \le 0.8025) = \Pr(11.95 \le X \le 16.05) = \Pr(12 \le X \le 16) = 0.7795$$

EXAMPLE *Large population, large sample: 60% hold a licence, $n = 200$* NORMAL APPROXIMATION

$\hat{P}$ is approximately normal with $\mu = p = 0.6$ and $\sigma = \sqrt{\dfrac{0.6(1-0.6)}{200}} = 0.0346$.

$$\Pr(\hat{P} > 0.65) = \texttt{normCdf(0.65,}\infty\texttt{,0.6,0.0346)} = 0.0745$$

Confidence interval for a proportion

The interval

$$\left(\hat{p} - z\sqrt{\tfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\tfrac{\hat{p}(1-\hat{p})}{n}}\right)$$

Margin of error ME $= z\,\sigma$, so the limits are $\hat{p} \pm$ ME and the width is 2 ME.

$z = 1.64$ for 90%, $z = 1.96$ for 95%, $z = 2.58$ for 99%.

Worked example

A sample of $n = 100$ gives $\hat{p} = 0.5$. The 95% interval is

$$0.5 \pm 1.96\sqrt{\tfrac{0.5(0.5)}{100}} = 0.5 \pm 0.098 \approx (0.40, 0.60).$$

On the TI-Nspire: `Menu 6-6-5` with $x = n\hat{p} = 50$, $n = 100$, C level 0.95.

What the interval means

A 95% confidence interval means that across many samples, 95% of the intervals constructed this way contain the true proportion p. It is not a statement about a single fixed p being random.

07 Useful Tips

Identities, formulas and exam reminders

The smaller results that save marks: identities, laws and reminders.

Angle between two lines

$\theta = \pm(\tan^{-1}(m_1) - \tan^{-1}(m_2))$

taking $m_1 > m_2$

Complementary shifts

- $\sin(x + \frac{\pi}{2}) = \cos x$
- $\sin(x - \frac{\pi}{2}) = -\sin x$
- $\sin(\frac{\pi}{2} - \theta) = \cos\theta$, $\cos(\frac{\pi}{2} - \theta) = \sin\theta$
- $\cos(\frac{\pi}{2} + \theta) = -\sin\theta$

Log laws

- $\log_b(MN) = \log_b M + \log_b N$
- $\log_b \frac{M}{N} = \log_b M - \log_b N$
- $\log_b(M^k) = k \log_b M$
- $\log_b 1 = 0,\ \log_b b = 1,\ b^{\log_b k} = k$

Binomial and hypergeometric

- **Binomial**, with replacement: independent trials, constant p.
- **Hypergeometric**, without replacement: each trial depends on the last, and p changes.

Remainder theorem

$P(x) \div (\beta x + \alpha) \Rightarrow r = P\left(-\frac{\alpha}{\beta}\right)$

e.g. $3x^3 + 2x^2 + 1$ divided by $(2x + 1)$ gives $r = \frac{9}{8}$

Change of base

$$\log_a x = \frac{\log_b x}{\log_b a} = \frac{\log_e x}{\log_e a}$$

Function tests, odd and even

- A relation is a function if it passes the vertical line test.
- It is one to one if it passes the horizontal line test.
- **Even**: $f(-x) = f(x)$, symmetric in the y axis.
- **Odd**: $f(-x) = -f(x)$, rotational symmetry.

Index laws

- $a^m a^n = a^{m+n}$, $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$, $(ab)^m = a^m b^m$
- $a^{-m} = \dfrac{1}{a^m}$, $a^0 = 1$
- $a^{m/n} = \sqrt[n]{a^m}$

Simultaneous equations

For $ax + by = c$ and $a'x + b'y = c'$:

- **One solution** $\frac{a}{a'} \neq \frac{b}{b'}$
- **Infinitely many** $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$
- **None** $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$

Domain of a sum or product

$(f + g)$ and $(f \cdot g)$ take the common domain of both.

$f = \log_e(3 - x)$ needs $x < 3$; $g = \sqrt{x + 4}$ needs $x \geq -4$; so the domain is $[-4, 3)$.

Derivative of an inverse function

If $f(a) = b$, then $f^{-1}(b) = a$: the point (a, b) on f corresponds to (b, a) on f^{-1}.
At those two corresponding points the gradients are **reciprocals** of each other:

$$f'(a) = \frac{1}{(f^{-1})'(b)} \qquad \text{equivalently} \qquad \left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$$

Measurement

Trapezium area	$\frac{1}{2}(a+b)h$	Cone volume	$\frac{1}{3}\pi r^2 h$
Cylinder surface	$2\pi rh$	Pyramid volume	$\frac{1}{3}Ah$
Cylinder volume	$\pi r^2 h$	Sphere volume	$\frac{4}{3}\pi r^3$
Triangle area	$\frac{1}{2}bc\sin A$		

Probability key words

at least means $\geq$; **at most** means $\leq$; **no more than** means $\leq$; **no less than** means $\geq$.

And above all

Check every answer, and do not make careless mistakes.

08 TI-Nspire

Calculator menu reference

Before the exam

Create 6 problem pages (`Ctrl Shift`, `Ctrl Menu 8`): Problems 1 to 5 for Section B, Problem 6 for Section A. Always Define, and move between tabs with `Ctrl left/right`.

Algebra

`Menu 3-1` Solve(Equation, x)

`Menu 3-7-1` Solve simultaneous

`Menu 3-3` Expand

`Menu 3-5` Complete the square

Calculus

`Shift -` Differentiate `Shift +` Integrate

`Menu 4-9` Tangent line

`Menu 4-A` Normal line

`fMin()` `fMax()`

Graphs

`Menu 6-1` x intercept `6-2 / 6-3` Min / Max

`Menu 6-4` Intersection `6-5` Inflection

`Menu 6-7` Integral (area to the axis)

`Menu 6-8` Area between two curves

`Menu 5-1` Trace `Menu 4` Zoom

Probability

`Menu 5-5-1 / 2` normalPdf / Cdf

`Menu 5-5-3` invNorm (area from the left)

`Menu 5-5-A / B` binomPdf / Cdf

`Menu 5-5-D` invBinom

`Menu 6-6-5` Confidence interval

Vi du nhu the nay chang han !!!